Normative vs. Positive Models: Choice under Uncertainty

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• Modern theory of choice dates from von Neumann and Morgenstern (1944)
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• precursors date from 18th century (D. Bernoulli)
• In von Neumann-Morgenstern model, choices made according to expected utility (EU) maximization
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\[ u : X \rightarrow \mathbb{R} \]

options
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$$u : X \rightarrow \mathbb{R}$$

- choose option that maximizes $\sum_{x \in X} p_x u(x)$, $p_x = \text{prob of } x$
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\[ \text{options} \]

– choose option that maximizes \[ \sum_{x \in X} p_x u(x) \quad p_x = \text{prob of } x \]

• justification is *axiomatic*
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\[ u : X \rightarrow \mathbb{R} \]

- choose option that maximizes \( \sum_{x \in X} p_x u(x) \) \( p_x = \text{prob of } x \)

**justification is axiomatic**

- rather than just *assuming* EU maximization, vN-M showed that if decision maker (DM) satisfies basic, rather compelling assumptions, must act *as though* maximizing EU
• One virtue of axiomatic approach:
• One virtue of axiomatic approach:
  – can understand complicated (and seemingly arbitrary) phenomenon (e.g., EU maximization) as implication of simple and less-arbitrary assumptions
vN-M model both normative and positive
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• started as *normative*
vN-M model both normative and positive

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  – how *should* rational DM behave under conditions of uncertainty
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  – explained much investment behavior
vN-M model both normative and positive

• started as *normative*
  – how *should* rational DM behave under conditions of uncertainty

• turned out to be *positive* too
  – explained much investment behavior
  – explained insurance markets well
• Of course, not all people are rational (nobody fully rational)
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  – and even fairly rational people make mistakes
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• but if mistakes are *random*
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  – situations where theory fails systematically
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  – so rational model works well
• Unfortunately, some *anomalies* discovered
  – situations where theory fails systematically
  – will discuss paradoxes of Allais, Ellsberg, Kahneman-Tversky
vN-M model
\( vN-M \text{ model} \)

- possible outcomes \( X = \{ x_1, \ldots, x_n \} \)
νN-M model

• possible outcomes \( X = \{x_1, \ldots, x_n\} \)
  – today, think of outcome as monetary
\( \nu N-M \) model

- possible outcomes \( X = \{ x_1, \ldots, x_n \} \)
  
  - today, think of outcome as \textit{monetary}
  
  - finite number of possibilities
\emph{vN-M model}

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  - today, think of outcome as \textit{monetary}
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- lottery: \textit{probability distribution} over outcomes
\(vN-M\) model

- possible outcomes \(X = \{x_1, \ldots, x_n\}\)
  - today, think of outcome as *monetary*
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  - \(\ell = \{p_1, \ldots, p_n\}\), \(p_i = \text{prob of } x_i\)
vN-M model

• possible outcomes \( X = \{x_1, \ldots, x_n\} \)
  
  – today, think of outcome as monetary
  – finite number of possibilities

• lottery: probability distribution over outcomes
  
  – \( \ell = \{p_1, \ldots, p_n\} \), \( p_i = \text{prob of } x_i \)

• DM chooses among lotteries
• DM has preferences over lotteries
• DM has preferences over lotteries
  \( \ell \succeq \ell' \)  
  DM (weakly) prefers \( \ell \) to \( \ell' \)
• DM has preferences over lotteries

\[ \ell \succeq \ell' \quad \text{DM (weakly) prefers } \ell \text{ to } \ell' \]

\[ \ell \succ \ell' \quad \text{DM strictly prefers } \ell \text{ to } \ell' \]
• DM has preferences over lotteries
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  \( \ell \sim \ell' \)  DM indifferent between \( \ell \) and \( \ell' \)
• DM has preferences over lotteries
  \( l \succeq l' \)  DM (weakly) prefers \( l \) to \( l' \)
  \( l > l' \)  DM strictly prefers \( l \) to \( l' \)
  \( l \sim l' \)  DM indifferent between \( l \) and \( l' \)
• vN-M imposed *axioms* on preferences
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• vN-M imposed \textit{axioms} on preferences

(1) \succeq \text{ satisfies}
  - \textit{reflexivity}: \( l \succeq l \)
  - \textit{completeness}: for any \( l \) and \( l' \), either \( l \succeq l' \) or \( l' \succeq l \)
• DM has preferences over lotteries
  \( \ell \succeq \ell' \)  DM (weakly) prefers \( \ell \) to \( \ell' \)
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  - *reflexivity*: \( \ell \succeq \ell \)
  - *completeness*: for any \( \ell \) and \( \ell' \), either \( \ell \succeq \ell' \) or \( \ell' \succeq \ell \)
  - *transitivity*: if \( \ell \succeq \ell' \) and \( \ell' \succeq \ell'' \), then \( \ell \succeq \ell'' \)
• DM has preferences over lotteries
  \( l \succeq l' \)  DM (weakly) prefers \( l \) to \( l' \)
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• vN-M imposed axioms on preferences

(1) \( \succeq \) satisfies
  - reflexivity: \( l \succeq l \)
  - completeness: for any \( l \) and \( l' \), either \( l \succeq l' \) or \( l' \succeq l \)
  - transitivity: if \( l \succeq l' \) and \( l' \succeq l'' \), then \( l \succeq l'' \)

• from (1), can assume \( x_1 \succ x_2 \succ \ldots \succ x_n \) (labeling)
(2) $\simeq$ satisfies continuity:
(2) \( \preceq \) satisfies \textit{continuity}:

- for any \( \ell \) there exists probability \( p \) such that

\[
\ell \sim \begin{cases} 
p \quad & x_1 \\
1-p \quad & x_n
\end{cases}
\]
(2) \(\prec\) satisfies \textit{continuity}:

- for any \(\ell\) there exists probability \(p\) such that

\[
\ell \sim \begin{cases}
\text{p} & x_1 \\
1 - p & x_n
\end{cases}
\]

(3) \(\succ\) satisfies \textit{monotonicity}:

\[
\begin{cases}
p & x_1 \\
1 - p & x_n
\end{cases} \sim \begin{cases}
p' & x_1 \\
1 - p' & x_n
\end{cases}
\]
(2) \(\succeq\) satisfies continuity:

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\[
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\]

(3) \(\succeq\) satisfies monotonicity:

\[
\begin{cases}
p & x_1 \\
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\end{cases} \succeq \begin{cases}
p' & x_1 \\
1-p' & x_n
\end{cases}
\]

if and only if \(p \geq p'\)
(4) $\sim$ satisfies *independence*:
(4) \(\succeq\) satisfies *independence*:
  - most controversial axiom
(4) \( \succcurlyeq \) satisfies *independence*:

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- suppose \( \ell \succcurlyeq \ell' \)
(4) \( \succeq \) satisfies *independence*:

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- suppose \( \ell \succeq \ell' \)

- then for all \( p \) and \( \hat{\ell} \)
(4) \( \succeq \) satisfies *independence*:
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- suppose \( \ell \succeq \ell' \)
- then for all \( p \) and \( \hat{\ell} \)

- only difference between two lotteries is:
  - on right side, \( \ell \) replaced by \( \ell' \)
Proposition (vN-M): if \( \succsim \) satisfies axioms (1) - (4) then
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if and only if

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\sum p_i u(x_i) \geq \sum p_i' u(x_i)
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Proposition (vN-M): if $\succeq$ satisfies axioms (1) - (4) then there exists $u: \{x_1, \ldots, x_n\} \rightarrow \mathbb{R}$ such that

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if and only if

$$\sum p_i u(x_i) \geq \sum p'_i u(x_i)$$

- so DM chooses lottery that maximizes EU
Proof:

- let $u(x_1) = 1, \ u(x_n) = 0$
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- let $u(x_1) = 1$, $u(x_n) = 0$
- from continuity, for every $x_i$, there exists probability $u(x_i)$ such that
\{ p_1, \ldots, p_n \} \sim \{ p'_1, \ldots, p'_n \}
\{p_1, \ldots, p_n\} \sim \{p'_1, \ldots, p'_n\}

\leftrightarrow

\text{independence}
\[ \{ p_1, \ldots, p_n \} \sim \{ p'_1, \ldots, p'_n \} \]

\[\leftrightarrow\]

\[\sum p_i u(x_i) \leftrightarrow x_i \sim \sum p'_i u(x_i) \leftrightarrow x'_i\]

\[\leftrightarrow\]

\[1 - \sum p_i u(x_i) \leftrightarrow x_n \sim 1 - \sum p'_i u(x_i) \leftrightarrow x'_n\]

independence

addition and multiplication
\[ \{ p_1, \ldots, p_n \} \sim \{ p'_1, \ldots, p'_n \} \]

\[ x \leftrightarrow \]

\[ \sum_{i=1}^n p_i u(x_i) \geq \sum_{i=1}^n p'_i u(x_i) \]

\[ \leftrightarrow \]

\[ \text{independence} \]

\[ \leftrightarrow \]

\[ \text{addition and multiplication} \]

\[ \leftrightarrow \]

\[ \text{monotonicity} \]
• DM is *risk averse* if
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  - prefers \( px_i + (1 - p)x_j \)
    to
  lottery with
    probability \( p \) of \( x_i \)
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  - i.e., prefers “sure thing” to lottery
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• risk aversion explains insurance market
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• risk aversion explains insurance market
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  - e.g., there’s a small chance your house may burn down
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- risk aversion \( \leftrightarrow \) utility function \( u \) concave
If monetary outcomes are unboundedly large
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- then \( u \) must be concave eventually
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To see this, consider the following lottery:
- probability $1/2$ of £1
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- probability \( 1/2 \) of £1
- probability \( 1/4 \) of £2
- probability \( 1/8 \) of £4
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• to see this, consider the following lottery:
  - probability \( 1/2 \) of £1
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  - probability \( 1/2^{n+1} \) of £2\(^n\)
• If monetary outcomes are unboundedly large
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• to see this, consider the following lottery:
  - probability \( 1/2 \) of £1
  - probability \( 1/4 \) of £2
  - probability \( 1/8 \) of £4
  - probability \( 1/2^{n+1} \) of £\( 2^n \)
• How much would DM be willing to pay for lottery?
• If monetary outcomes are unboundedly large
  - then \( u \) must be concave eventually
• to see this, consider the following lottery:
  - probability \( \frac{1}{2} \) of £1
  - probability \( \frac{1}{4} \) of £2
  - probability \( \frac{1}{8} \) of £4
  - probability \( \frac{1}{2^{n+1}} \) of £\( 2^n \)
• How much would DM be willing to pay for lottery?
  - expected value:
    \[
    \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 4 + \ldots
    = \infty!
    \]
• If monetary outcomes are unboundedly large
  - then $u$ must be concave eventually
• to see this, consider the following lottery:
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    \]
    \[= \infty!
    \]
  - but no one would be willing to pay $\infty$
If monetary outcomes are unboundedly large, then \( u \) must be concave eventually.

to see this, consider the following lottery:

- probability \(1/2\) of £1
- probability \(1/4\) of £2
- probability \(1/8\) of £4
- probability \(1/2^{n+1}\) of £2\(^n\)

How much would DM be willing to pay for lottery?

- expected value:

\[
\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 4 + \ldots
\]

= \(\infty\)!

- but no one would be willing to pay \(\infty\)
- so DM’s utility function must be concave eventually
• Example called St. Petersburg Paradox
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• resolved by Bernoulli (1738)
• vN-M model applies very widely
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• but some well-documented violations
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• but some well-documented violations
• one pointed out by Allais (1953)
• Suppose DM offered choice between
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  – £1 million for sure (A)
• Suppose DM offered choice between
  
  – £1 million for sure  (A)

and
Suppose DM offered choice between

- £1 million for sure (A)

and

- lottery

\[
\begin{array}{c}
\text{0.01} \downarrow \\
$1m \\
\text{0.89} \downarrow \\
$5m \\
\text{0.10} \downarrow
\end{array}
\]
• Suppose DM offered choice between

  – £1 million for sure (A)

and

  – lottery

    \[
    \begin{array}{c}
    \text{£5m} \\
    \text{£1m} \\
    0
    \end{array}
    \]

\[
\begin{array}{c}
\cdot10 \\
\cdot89 \\
\cdot01
\end{array}
\]

• most people choose A
• Now, suppose DM offered choice between

\[ \begin{array}{c}
\text{.11} \\
\text{.89}
\end{array} \]

\[ \begin{array}{c}
\text{£1m} \\
0
\end{array} \]

(C)
• Now, suppose DM offered choice between

\[ \begin{array}{c}
\begin{array}{c}
\text{.11} \\
\text{.89}
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\£1m \\
0
\end{array}
\end{array} \]  (C)

and
• Now, suppose DM offered choice between

\[
\begin{array}{c}
\text{.11} & £1\text{m} \\
\text{.90} & 0 \quad (C)
\end{array}
\]

and

\[
\begin{array}{c}
\text{.10} & £5\text{m} \\
\text{.90} & 0 \quad (D)
\end{array}
\]
• Now, suppose DM offered choice between

\[ \begin{array}{c}
\begin{array}{c}
\text{£1m} \\
0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
.11 \\
.89
\end{array}
\end{array} \]

and

\[ \begin{array}{c}
\begin{array}{c}
\text{£5m} \\
0
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
.10 \\
.90
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\text{(C)} \\
\text{(D)}
\end{array}
\end{array} \]

• most people choose D
• Now, suppose DM offered choice between

\[
\begin{align*}
  &0.11 \rightarrow \text{£1m} \rightarrow (C) \\
  &0.89 \rightarrow 0 \\
\end{align*}
\]

and

\[
\begin{align*}
  &0.10 \rightarrow \text{£5m} \rightarrow (D) \\
  &0.90 \rightarrow 0 \\
\end{align*}
\]

• most people choose D
• but choices A and D together violate EU!
• A can be rewritten as

\[
\begin{align*}
\text{\£1m} & \quad \text{.89} \\
\text{\£1m} & \quad \text{.71}
\end{align*}
\]
• A can be rewritten as

• B can be rewritten
• A can be rewritten as

• B can be rewritten

• but if $A \succ B$, then independence axiom implies
- A can be rewritten as
  \[ \gamma \]

- B can be rewritten
  \[ \gamma \]

- but if \( A \succ B \), then independence axiom implies
  \[ \gamma \]

- so \( C \succ D \)
• So far, have been taking probabilities as “objective”
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  – but, in reality, usually are not (except in casinos, etc.)
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• if buy a share of IBM
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• if buy a share of IBM
  – could go up by $10
  – could go down by $7
• So far, have been taking probabilities as “objective”
  – but, in reality, usually are not (except in casinos, etc.)
• if buy a share of IBM
  – could go up by $10
  – could go down by $7
  – could stay the same
• So far, have been taking probabilities as “objective”
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• if buy a share of IBM
  – could go up by $10
  – could go down by $7
  – could stay the same
  – probabilities of these events not “prescribed”-- they are subjective
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  – could go down by $7
  – could stay the same
  – probabilities of these events not “prescribed”-- they are subjective

• Savage (1954) reformulates vN-M axioms so that apply to case of subjective probability
• Independence axiom becomes:
Independence axiom becomes:

- if $\ell \succeq \ell'$
• Independence axiom becomes:

- if $\ell \succeq \ell'$

- then for all events $E$ and all $\hat{\ell}$
Proposition (Savage): if $\succsim$ satisfies Savage’s axioms
Proposition (Savage): if \( \succeq \) satisfies Savage’s axioms

- then there exists a probability distribution \( p(\bullet) \) and utility function \( u : X \to \mathbb{R} \) such that
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  - where \( x_{\ell E} = \text{outcome of lottery } \ell \text{ in state } E \)
  - \( x_{\ell' E} = \text{outcome of lottery } \ell' \text{ in state } E \)
• Famous violation of Savage’s axioms due to D. Ellsberg
• Famous violation of Savage’s axioms due to D. Ellsberg
• same Ellsberg who leaked “Pentagon Paper” to press
Closed box containing 90 colored balls

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<thead>
<tr>
<th></th>
<th>30</th>
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<tbody>
<tr>
<td></td>
<td>red</td>
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<tr>
<td>$\ell_1$</td>
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Closed box containing 90 colored balls

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<th>30 (red)</th>
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- most people prefer $l_1$ to $l_2$
Closed box containing 90 colored balls

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- most people prefer $\ell_1$ to $\ell_2$
- most people prefer $\ell_4$ to $\ell_3$
Closed box containing 90 colored balls

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- most people prefer $l_1$ to $l_2$
- most people prefer $l_4$ to $l_3$
- violates Savage
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- most people prefer $\ell_1$ to $\ell_2$
- most people prefer $\ell_4$ to $\ell_3$
- violates Savage
  - $\ell_1 \succ \ell_2 \rightarrow p(\text{red}) > p(\text{black})$
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- most people prefer \(\ell_1\) to \(\ell_2\)
- most people prefer \(\ell_4\) to \(\ell_3\)
- violates Savage
  - \(\ell_1 \succ \ell_2 \rightarrow p(\text{red}) > p(\text{black})\)
  - \(\ell_4 \succ \ell_3 \rightarrow p(\text{black}) + p(\text{yellow}) > p(\text{red}) + p(\text{yellow})\)
Kahneman-Tversky (1981)
Kahneman-Tversky (1981)
• casts doubt on whether can represent lottery unambiguously as \( \ell = (p_1, \ldots, p_n) \)
600 citizens exposed to deadly disease
600 citizens exposed to deadly disease

- treatment A: saves 200 lives
600 citizens exposed to deadly disease

- treatment A: saves 200 lives
- treatment B:
600 citizens exposed to deadly disease

- **treatment A**: saves 200 lives
- **treatment B**:
  
  ![Diagram]

  - 1/3 saves 600 lives
  - 2/3 nobody saved

- most people choose A over B
600 citizens exposed to deadly disease

- treatment A: saves 200 lives
- treatment B:
  - 1/3: 600 saved
  - 2/3: nobody saved

  - most people choose A over B

- treatment C: 400 die
600 citizens exposed to deadly disease

- treatment A: saves 200 lives
- treatment B:
  - 1/3: 600 saved
  - 2/3: nobody saved

  - most people choose A over B

- treatment C: 400 die
- treatment D:
  - 1/3: nobody dies
  - 2/3
    - 1/3: nobody dies
    - 2/3: 600 die
600 citizens exposed to deadly disease

- treatment A: saves 200 lives
- treatment B:
  - 1/3: 600 saved
  - 2/3: nobody saved

  - most people choose A over B

- treatment C: 400 die

- treatment D:
  - 1/3: nobody dies
  - 2/3:
    - 1/3: nobody dies
    - 2/3: 600 die

  - most people choose D over C
600 citizens exposed to deadly disease

- treatment A: saves 200 lives
- treatment B:
  - most people choose A over B
- treatment C: 400 die
- treatment D:
  - most people choose D over C
- but A equivalent to C, B equivalent to D!
• Have shown you 3 “anomalies”
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  – Allais
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  – Allais
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• there are about 8 or 9 more
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• by contrast in early days of decision theory, just one model
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• there are about 8 or 9 more
  – theoretical problem
  – there is a model that accounts for each of the dozen problems
  – but that means there are 12 models
• by contrast in early days of decision theory, just one model
  – challenge: to unify the 12